ON THE J-FLOW IN HIGHER DIMENSIONS AND THE LOWER BOUNDEDNESS OF THE MABUCHI ENERGY

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Abstract. The J-flow is a parabolic flow on Kähler manifolds. It was defined by Donaldson in the setting of moment maps and by Chen as the gradient flow of the J-functional appearing in his formula for the Mabuchi energy. It is shown here that under a certain condition on the initial data, the J-flow converges to a critical metric. This is a generalization to higher dimensions of the author's previous work on Kähler surfaces. A corollary of this is the lower boundedness of the Mabuchi energy on Kähler classes satisfying a certain inequality when the first Chern class of the manifold is negative.

1. Introduction

The J-flow is a parabolic flow of potentials on Kähler manifolds with two Kähler classes. It was defined by Donaldson [D1] in the setting of moment maps and by Chen [C1] as the gradient flow for the J-functional appearing in his [C1] formula for the Mabuchi energy [Ma]. Chen [C2] proved long-time existence of the flow for any smooth initial data, and proved a convergence result in the case of non-negative bisectional curvature.

The J-flow is defined as follows. Let (M, ω) be a compact Kähler manifold of complex dimension n and let χ_0 be another Kähler form on M. Let $\mathcal{H}$ be the space of Kähler potentials

$$\mathcal{H} = \{\phi \in C^\infty(M) \mid \chi_\phi = \chi_0 + \frac{\sqrt{-1}}{2}\partial\overline{\partial}\phi > 0\}.$$

The J-flow is the flow in $\mathcal{H}$ given by

$$\begin{cases} \dfrac{\partial \phi_t}{\partial t} = c - \dfrac{\omega \wedge \chi_{\phi_t}^{n-1}}{\chi_{\phi_t}^n} \\ \phi_0 = 0, \end{cases} \tag{1.1}$$

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where c is the constant given by

$$c = \frac{\int_M \omega \wedge \chi_0^{n-1}}{\int_M \chi_0^n}.$$

A critical point of the J-flow gives a Kähler metric χ satisfying

$$\omega \wedge \chi^{n-1} = c\chi^n. \tag{1.2}$$

Donaldson [D1] showed that a necessary condition for a solution to (1.2) in $[\chi_0]$ is that $[nc\chi_0 - \omega]$ be a positive class. He remarked that a natural conjecture would be that this condition be sufficient. Chen [C1] proved this result if $n = 2$, without using the J-flow. In [W1], the author gave an alternative proof by showing that for $n = 2$, the J-flow converges in C^∞ to a critical metric under the condition $nc\chi_0 - \omega > 0$.

We generalize this as follows.

Main Theorem. *If the Kähler metrics ω and χ_0 satisfy*

$$nc\chi_0 - (n-1)\omega > 0,$$

then the J-flow (1.1) converges in C^∞ to a smooth critical metric.

This shows that (1.2) has a solution in $[\chi_0]$ under the condition

$$nc[\chi_0] - (n-1)[\omega] > 0.$$

Recall that the Mabuchi energy is a functional on $\mathcal{H}$ defined by

$$M_{\chi_0}(\phi) = -\int_0^1 \int_M \frac{\partial \phi_t}{\partial t}(R_t - \underline{R}) \frac{\chi_{\phi_t}^n}{n!} dt,$$

where $\{\phi_t\}_{0 \le t \le 1}$ is a path in $\mathcal{H}$ between 0 and ϕ, R_t is the scalar curvature of χ_{ϕ_t} and $\underline{R}$ is the average of the scalar curvature. The critical points of this functional are metrics of constant scalar curvature.

If $c_1(M) < 0$ then there exists a Kähler-Einstein metric in the class $-c_1(M)$ ([Y1], [Au]). It follows easily that the Mabuchi energy is bounded below in this class. Also, the result of Donaldson [D3] implies that M is asymptotically Chow stable with respect to the canonical bundle. More generally, for any class, it is expected that the lower boundedness of the Mabuchi energy should be equivalent to some notion of semistability in the sense of geometric invariant theory (see [Y2], [T2, T3], [PS] and [D4]).

Chen [C1] shows that if $c_1(M)$ is negative with $-\omega \in c_1(M)$ and if there is a solution of (1.2) in $[\chi_0]$, then the Mabuchi energy is bounded below *in the class* $[\chi_0]$. This suggests that (1.2) is related to how a change of polarization affects the condition of stability of a manifold. We immediately have the following corollary of the main theorem, generalizing the result of Chen for dimension $n = 2$.

Corollary. *Let M be a compact Kähler manifold with $c_1(M) < 0$. If the Kähler class $[\chi_0]$ satisfies the inequality*

$$-n\frac{c_1(M) \cdot [\chi_0]^{n-1}}{[\chi_0]^n}[\chi_0] + (n-1)c_1(M) > 0, \tag{1.3}$$

then the Mabuchi energy is bounded below on $[\chi_0]$.

Note that if $[\chi_0] = [-c_1(M)]$ then the inequality is more than adequately satisfied, and so the set of $[\chi_0]$ satisfying the condition is a reasonably large open set containing the canonical class.

In section 2, we prove a second order estimate of ϕ in terms of ϕ itself, and in section 3, we prove the zero order estimate and complete the proof of the main theorem. The techniques used are generalizations of those given in [W1], and we will refer the reader to that paper for some of the calculations and arguments.

In the following, $C_0, C_1, C_2, \ldots$ will denote constants depending only on the initial data.

2. Second order estimate

We use the maximum principle to prove the following estimate on the second derivatives on ϕ.

Theorem 2.1 *Suppose that*

$$nc\chi_0 - (n-1)\omega > 0.$$

Let $\phi = \phi_t$ be a solution of the J-flow (1.1) on $[0, \infty)$. Then there exist constants $A > 0$ and $C > 0$ depending only on the initial data such that for any time $t \geq 0$, $\chi = \chi_{\phi_t}$ satisfies

$$\Lambda_\omega \chi \leq Ce^{A(\phi - \inf_{M\times[0,t]} \phi)}. \tag{2.1}$$

Proof We will assume that ω has been scaled so that $c = 1/n$. Choose $0 < \epsilon < 1/(n+1)$ to be sufficiently small so that

$$\chi_0 \geq (n - 1 + (n+1)\epsilon)\omega. \tag{2.2}$$

We will use the same notation as in [W1]. In particular, the operator $\tilde{\triangle}$ acts on functions f by

$$\tilde{\triangle} f = \frac{1}{n} h^{k\bar{l}} \partial_k \partial_{\bar{l}} f, \qquad \text{where} \qquad h^{k\bar{l}} = \chi^{k\bar{j}} \chi^{i\bar{l}} g_{i\bar{j}}.$$

We calculate the evolution of $(\log(\Lambda_\omega \chi) - A\phi)$, where A is a constant to be determined. From [W1], we have

$$\begin{aligned}
&(\tilde{\triangle} - \frac{\partial}{\partial t})(\log(\Lambda_\omega \chi) - A\phi) \\
&\geq \frac{1}{n}(-C_0 h^{k\bar{l}} g_{k\bar{l}} - \frac{1}{\Lambda_\omega \chi} \chi^{k\bar{l}} R_{k\bar{l}} - 2A\chi^{i\bar{j}} g_{i\bar{j}} + A h^{k\bar{l}} \chi_{0\,k\bar{l}} + A) \\
&= \frac{1}{n}(-C_0 h^{k\bar{l}} g_{k\bar{l}} - \frac{1}{\Lambda_\omega \chi} \chi^{k\bar{l}} R_{k\bar{l}} - 2A\chi^{i\bar{j}} g_{i\bar{j}} + (1-\epsilon) A h^{k\bar{l}} \chi_{0\,k\bar{l}} \\
&\quad + \epsilon A h^{k\bar{l}} \chi_{0\,k\bar{l}} + A),
\end{aligned}$$

where C_0 is a lower bound for the bisectional curvature of ω, and $R_{k\bar{l}}$ is the Ricci curvature tensor of ω. Recall from (2.4) in [W1], that χ is uniformly bounded away from zero. Hence we can choose A to be large enough so that

$$\epsilon A h^{k\bar{l}} \chi_{0\,k\bar{l}} \geq C_0 h^{k\bar{l}} g_{k\bar{l}} + \frac{1}{\Lambda_\omega \chi} \chi^{k\bar{l}} R_{k\bar{l}}.$$

Now fix a time $t > 0$. There is a point (x_0, t_0) in $M \times [0, t]$ at which the maximum of $(\log(\Lambda_\omega \chi) - A\phi)$ is achieved. We may assume that $t_0 > 0$. Then at this point (x_0, t_0), we have

$$1 + (1-\epsilon) h^{k\bar{l}} \chi_{0\,k\bar{l}} - 2\chi^{i\bar{j}} g_{i\bar{j}} \leq 0.$$

From (2.2), we get

$$1 + (n - 1 + \epsilon) h^{k\bar{l}} g_{k\bar{l}} - 2\chi^{i\bar{j}} g_{i\bar{j}} \leq 0.$$

We will compute in normal coordinates at x_0 for ω in which χ is diagonal and has eigenvalues $\lambda_1 \leq \ldots \leq \lambda_n$. The above inequality becomes

$$1 + (n - 1 + \epsilon) \sum_{i=1}^{n} \frac{1}{\lambda_i^2} - 2 \sum_{i=1}^{n} \frac{1}{\lambda_i} \leq 0. \tag{2.3}$$

We claim that (2.3) gives an upper bound for the λ_i. To see this, complete the square to obtain

$$\sum_{i=1}^{n} \left(\frac{1}{\sqrt{n-1+\epsilon}} - \frac{\sqrt{n-1+\epsilon}}{\lambda_i} \right)^2 \leq \frac{n}{n-1+\epsilon} - 1.$$

Hence, for $i = 1, \ldots, n$,

$$\frac{1}{\sqrt{n-1+\epsilon}} - \frac{\sqrt{n-1+\epsilon}}{\lambda_i} \leq \frac{\sqrt{1-\epsilon}}{\sqrt{n-1+\epsilon}},$$

from which we obtain the upper bound

$$\lambda_i \leq \frac{n-1+\epsilon}{1-\sqrt{1-\epsilon}}.$$

Hence at the point (x_0, t_0), we have a constant C depending only on the initial data such that

$$\Lambda_\omega \chi \leq C.$$

Then, on $M \times [0, t]$,

$$\log(\Lambda_\omega \chi) - A\phi \leq \log C - A \inf_{M \times [0,t]} \phi.$$

Exponentiating gives

$$\Lambda_\omega \chi \leq C e^{A(\phi - \inf_{M \times [0,t]} \phi)},$$

completing the proof of the theorem.

3. Proof of the Main Theorem

We know from [C2] that the flow exists for all time. To prove the main theorem we need uniform estimates on ϕ_t and all of its derivatives. Given such estimates, the argument of section 5 of [W1], which is valid for any dimension, shows that ϕ_t converges in C^∞ to a smooth critical metric.

From Theorem 2.1, and standard parabolic methods, it suffices to have a uniform C^0 estimate on ϕ. We prove this below, generalizing the method of [W1], using the precise form of the estimate (2.1) and a Moser iteration type argument.

Theorem 3.1 *Suppose that*

$$nc\chi_0 - (n-1)\omega > 0.$$

Let $\phi = \phi_t$ be a solution of the J-flow (1.1) on $[0, \infty)$. Then there exists a constant $\tilde{C} > 0$ depending only on the initial data such that

$$\|\phi_t\|_{C^0} \leq \tilde{C}.$$

Proof We begin with a lemma.

Lemma 3.2 $0 \leq \sup_M \phi_t \leq -C_1 \inf_M \phi_t + C_2$.

Proof We will use the functional I_{χ_0} defined on $\mathcal{H}$ by

$$I_{\chi_0}(\phi) = \int_0^1 \int_M \frac{\partial \phi_t}{\partial t} \frac{\chi_{\phi_t}^n}{n!} dt, \tag{3.1}$$

for $\{\phi_t\}$ a path between 0 and ϕ (this is a well-known functional, see [D2] for example). Taking the path $\phi_t = t\phi$, we obtain the formula:

$$\begin{aligned} I_{\chi_0}(\phi) &= \frac{1}{n!} \int_0^1 \int_M \phi \, \chi_{t\phi}^n dt \\ &= \frac{1}{n!} \int_0^1 \int_M \phi \, (t\chi_\phi + (1-t)\chi_0)^n dt \\ &= \frac{1}{n!} \sum_{k=0}^n \left[\binom{n}{k} \int_0^1 t^k (1-t)^{n-k} dt \right] \int_M \phi \, \chi_\phi^k \wedge \chi_0^{n-k}. \end{aligned} \tag{3.2}$$

From (3.1), we see that $I(\phi_t) = 0$ along the flow. The first inequality then follows immediately, since the expression in the square brackets in (3.2) is a positive function of n and k. The second inequality follows from (3.2), the fact that $\triangle_\omega \phi_t > -\Lambda_\omega \chi_0$, and properties of the Green's function of ω.

From this lemma, it is sufficient to prove a lower bound for $\inf_M \phi_t$. If such a lower bound does not exist, then we can choose a sequence of times $t_i \to \infty$ such that

(i) $\inf_M \phi_{t_i} = \inf_{t \in [0, t_i]} \inf_M \phi_t$

(ii) $\inf_M \phi_{t_i} \to -\infty$.

We will find a contradiction. Set $B = A/(1-\delta)$ where A is the constant from (2.1), and let δ be a small positive constant to be determined later. Let

$$\psi_i = \phi_{t_i} - \sup_M \phi_{t_i},$$

and let $u = e^{-B\psi_i}$. We will show that u is uniformly bounded from above, which will give the contradiction. First, we have the following lemma.

Lemma 3.3 *For any $p \geq 1$,*

$$\int_M |\nabla u^{p/2}|^2 \frac{\omega^n}{n!} \leq C_3\, p\, \|u\|_{C^0}^{1-\delta} \int_M u^{p-(1-\delta)} \frac{\omega^n}{n!}. \tag{3.3}$$

Proof The proof is given for $n = 2$ in [W1], and since the same argument works for any dimension, we will not reproduce it here. Crucially, the proof uses the estimate (2.1).

We will use the notation

$$\|f\|_c = \left(\int_M |f|^c \frac{\omega^n}{n!} \right)^{1/c},$$

for $c > 0$. It is not a norm for $0 < c < 1$ but this fact is not important. The following lemma allows us to estimate the C^0 norm of u using a Moser iteration type method (compare to [Y1]).

Lemma 3.4 *If $u \geq 0$ satisfies the estimate (3.3) for all $p \geq 1$, then for some constant C' independent of u,*

$$\|u\|_{C^0} \leq C' \|u\|_\delta.$$

Proof For $\beta = n/(n-1)$, the Sobolev inequality for functions f on (M, ω) is

$$\|f\|_{2\beta}^2 \leq C_4(\|\nabla f\|_2^2 + \|f\|_2^2).$$

Applying this to $u^{p/2}$ and making use of (3.3) gives

$$\|u\|_{p\beta} \leq C_5^{1/p} p^{1/p} \|u\|_{C^0}^{\gamma/p} \|u\|_{p-\gamma}^{(p-\gamma)/p},$$

for $\gamma = 1 - \delta$. By replacing p with $p\beta + \gamma$ we obtain inductively

$$\|u\|_{p_k\beta} \leq C(k) \|u\|_{C^0}^{1-a(k)} \|u\|_{p-\gamma}^{a(k)},$$

where

$$\begin{aligned} p_k &= p\beta^k + \gamma(1+\beta+\beta^2+\cdots+\beta^{k-1}) \\ C(k) &= C_5^{(1+\beta+\cdots+\beta^k)/p_k} p_0^{\beta^k/p_k} p_1^{\beta^{k-1}/p_k} \cdots p_k^{1/p_k} \\ a(k) &= \frac{(p-\gamma)\beta^k}{p_k}. \end{aligned}$$

Set $p = 1$. Note that for some fixed l, $\beta^k \le p_k \le \beta^{k+l}$. It is easy to check that $C(k) \le C_6$ for some constant C_6. As k tends to infinity, $p_k \to \infty$, $a(k) \to a \in (0,1)$, and the required estimate follows immediately.

We can now finish the proof of Theorem 3.1. Since $u = e^{-B\psi_i}$ and ψ_i satisfies $\sup_M \psi_i = 0$ and

$$\chi_{0\,k\bar{l}} + \partial_k \partial_{\bar{l}} \psi_i \ge 0,$$

we can apply Proposition 2.1 of [T1] to get a bound on $\|u\|_\delta$ for δ small enough. This completes the proof of Theorem 3.1.

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